

Discrete symmetries of low-dimensional Dirac models: A selective review with a focus on condensed-matter realisations

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Abstract

The most fundamental characteristics of a physical system can often be deduced from its behaviour under discrete symmetry transformations such as time reversal, parity and chirality. Here we review basic symmetry properties of the relativistic quantum theories for free electrons in (2+1)- and (1+1)-dimensional spacetime. Additional flavour degrees of freedom are necessary to properly define symmetry operations in (2+1) dimensions and are generally present in physical realisations of such systems, e.g., in single sheets of graphite. We find that there exist two possibilities for defining any flavour-coupling discrete symmetry operation of the two-flavour (2+1)-dimensional Dirac theory. Physical implications of this duplicity are discussed.

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1. Introduction

The realisation of a two-dimensional (2D) quasi-relativistic electron system in single-layer graphene [1] and at the surface of topological insulators [2] has renewed interest in 2D versions of the Dirac equation [3, 4] for a free particle,

$$H_D \psi = i \partial_t \psi, \quad (1a)$$

with Hamiltonian

$$H_D(\mathbf{p}) = \mathbf{v} \cdot \mathbf{p} + \beta m. \quad (1b)$$

We summarise here some features arising due to the reduced dimensionality, focusing especially on symmetry properties. Comparison is made between previously

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considered (2+1)-dimensional quantum electrodynamics (QED₃) [5–7] and quasi-relativistic condensed-matter systems. [8–13]

1.1. Dirac Equation: Basics We employ the general expression for the Dirac Hamiltonian given in Eq. (1b) in terms of the velocity operator \mathbf{v} and the matrix β valid in any spatial dimensions $d \geq 1$. Our notation emphasizes the fact that the matrices α ($\equiv \mathbf{v}$) commonly used for the Dirac equation in (3+1)D have the physical interpretation of being the operator for the velocity of a free particle. While the explicit form of these matrix operators depends on the dimension, the structure of the Hamiltonian H_D does not. As the Dirac Hamiltonian for a free particle commutes with momentum \mathbf{p} , we can choose plane waves as basis in real space and represent \mathbf{p} as an ordinary vector in \mathbb{R}^d . In the following, we adhere to this convention of condensed-matter physics, which implies that only the constant matrices \mathbf{v} and β are quantum-mechanical operators. We use units where the Planck constant $\hbar = 1$ and also $|\mathbf{v}|^2 = 1$.

The spectrum of the Dirac Hamiltonian (1b) has two branches, one at positive and one at negative energies. These are given by

$$E_{\pm}(\mathbf{p}) = \pm \sqrt{|\mathbf{p}|^2 + m^2} \quad (2)$$

independent of the number d of spatial dimensions. The eigenstates in position representation have the general form

$$\psi_{\mathbf{p}}^{\pm}(\mathbf{r}) = e^{i\mathbf{p}\cdot\mathbf{r}} \chi_{\mathbf{p}}^{\pm} \quad (3)$$

In (3+1)D, the $\chi_{\mathbf{p}}^{\pm}$ are four-component spinors that encode the energy-band index and the Dirac particle’s intrinsic angular-momentum (spin) vector, where the corresponding spin operator is defined as

$$\mathbf{S} = \frac{1}{4i} \mathbf{v} \times \mathbf{v}. \quad (4)$$

Here the symbol ‘ \times ’ denotes the cross product, defined formally as for vectors in \mathbb{R}^3 .

The relativistically covariant form of the Dirac equation is obtained by multiplying Eq. (1a) by β and re-arranging to exhibit derivatives w.r.t. time and spatial coordinates in a unified fashion. Using the definitions $\gamma^0 = \beta$, $\gamma^j = \beta v_j$ with $j = 1, \dots, d$, as well as the components $p_{\mu} = i\partial_{x^{\mu}}$ of the $(d + 1)$ -vector of momentum [14], one finds

$$\left(\gamma^{\mu} p_{\mu} - m \right) \psi = 0 \quad , \quad (5)$$

where we have used the convention that repeated indices are summed over.

1.2. Symmetries The symmetries of the Dirac equation include operations that reflect the properties of Minkowski space time. (Point group operations give rise to the Lorentz group, whereas the Poincaré group also includes translations in space and time.) These symmetries are also relevant for a classical (non-quantum) description of relativistic systems. On the other hand, in general we also have symmetries that

are present only in a quantum mechanical description of relativistic systems. Each of these categories include discrete and continuous symmetries.

In the following, we will focus mostly on the discrete symmetries that are specific for the quantum-mechanical Dirac equation. Among the spatial symmetries, we will concentrate on the nontrivial operations of parity (space inversion) and rotations by 2π . Moreover, we will consider the operation of time reversal (also called *reversal of the motion*). [15] The Hilbert space of the (single-particle) Dirac equation includes solutions with positive and negative energies corresponding to particles and holes (i.e., anti-particles). These are related by *particle-hole conjugation* and *energy-reflection symmetry*. [9, 14, 16–18] Lorentz invariance will not feature at all.

We continue this Section with a brief introduction to particular discrete symmetries and review their representation for the (3+1)D Dirac case.

1.2.1. Spatial Symmetries — in particular, Parity A spatial symmetry transformation g is represented by a unitary operator $\hat{P}(g)$ acting in the Hilbert space of the Hamiltonian H , where the set $\{g\}$ of symmetry transformations forms a group G . To describe the symmetry of an observable O under the symmetry transformation g we use the short-hand notation $g O \equiv \hat{P}(g) O \hat{P}^{-1}(g)$. When $\hat{P}(g)$ acts on the basis functions $\{| \nu \rangle\}$ of H , we can expand $\hat{P}(g) | \nu \rangle$ in terms of the basis functions $\{| \mu \rangle\}$,

$$\hat{P}(g) | \nu \rangle = \sum_{\mu} \mathcal{D}(g)_{\mu\nu} | \mu \rangle, \quad (6)$$

so that the set of matrices $\{\mathcal{D}(g)\}$ forms a representation for G . Then the invariance of the Hamiltonian H under the symmetry transformation g implies [19]

$$\mathcal{D}(g) H(\mathbf{p}) \mathcal{D}^{-1}(g) = H(g \mathbf{p}). \quad (7)$$

This equation expresses the fact that we can view the effect of g from two equivalent perspectives: we may regard it as a unitary transformation of the basis functions $\{| \nu \rangle\}$ of H (“active” view). This is equivalent to applying g to momentum \mathbf{p} (and position \mathbf{r} , “passive” view). If g is a symmetry of H , both views must yield the same result. The invariance condition (7) holds generally for continuous symmetry transformations (such as rotations) as well as discrete operations (e.g., parity).

Equation (7) includes as a special case *parity* \mathcal{P} (i.e., space inversion). In $d = 3$ spatial dimensions, it is characterized via the relations

$$\mathcal{P} \mathbf{r} = -\mathbf{r}, \quad \mathcal{P} \mathbf{p} = -\mathbf{p} \quad (8a)$$

$$\mathcal{D}(\mathcal{P}) H(\mathbf{p}) \mathcal{D}^{-1}(\mathcal{P}) = H(\mathcal{P} \mathbf{p}) = H(-\mathbf{p}) \quad (8b)$$

For the Dirac Hamiltonian in (3+1)D, the representation matrix for the parity operation is

$$\mathcal{D}(\mathcal{P}) = \beta. \quad (9)$$

As to be expected, we have $\mathcal{D}(\mathcal{P}^2) = +\mathbb{1}$.

1.2.2. Rotations \mathcal{R} In (3+1)D, the Hamiltonian H_D commutes with the operator of total angular momentum $\mathbf{J} = \mathbf{L} + \mathbf{S}$, where $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ is the operator of orbital angular momentum, and the spin operator \mathbf{S} is given by Eq. (4). Total angular momentum \mathbf{J} is the generator for finite rotations. Rotations $\mathcal{R}_{\hat{n}}$ by 2π about an arbitrary axis \hat{n} maps the system onto itself with a minus sign

$$\mathcal{D}(\mathcal{R}_{\hat{n}}(2\pi)) = \exp(2\pi i \hat{n} \cdot \mathbf{J}) = -\mathbb{1} . \quad (10)$$

In that sense, the symmetry group becomes a double group which is common for half-integer spin systems [20].

1.2.3. Time Reversal Time reversal (TR) is an anti-unitary transformation and can be represented by the operator

$$\theta = \mathcal{T}\mathcal{K} \quad (11)$$

where \mathcal{K} denotes complex conjugation, and the operator \mathcal{T} is the unitary transformation that relates the time-reversed (i.e., complex-conjugated) basis spinors to the original basis [19, 21]. TR invariance requires that the Dirac equation has the same solution as the equation obtained by applying θ to both the basis functions and all tensors that represent physical quantities (such as momentum \mathbf{p}), giving

$$\mathcal{D}(\mathcal{T}) H^*(\mathbf{p}) \mathcal{D}(\mathcal{T})^{-1} = H(\theta \mathbf{p}) = H(-\mathbf{p}). \quad (12)$$

Here, $*$ denotes complex conjugation. The minus sign reflects the fact that momentum \mathbf{p} is odd under reversal of motion. This implies that θ reverts the velocity \mathbf{v} , $\mathcal{D}(\mathcal{T}) \mathbf{v}^* \mathcal{D}(\mathcal{T})^{-1} = -\mathbf{v}$, but does not change the mass term, $\mathcal{D}(\mathcal{T}) \beta^* \mathcal{D}(\mathcal{T})^{-1} = \beta$. For the Dirac Hamiltonian in (3+1)D in the standard representation [14] where only v_2 is imaginary, we have

$$\mathcal{D}(\mathcal{T}) = -i v_1 v_3 \equiv i \gamma^1 \gamma^3 , \quad (13)$$

which implies $\mathcal{D}(\theta^2) = -\mathbb{1}$. The remarkable property that the representation matrix for the time reversal operation squares to the *negative* of the unit matrix for a (3+1)D Dirac particle arises as a consequence of its intrinsic half-integer spin [22].

1.2.4. Particle-Hole Conjugation Particle-hole conjugation is also an anti-unitary operation [14, 16–18]. We can express it using the operator

$$\kappa = C\mathcal{K} , \quad (14)$$

with a unitary operator C . Particle-hole symmetry of a Hamiltonian holds if the relation

$$\mathcal{D}(C) H^*(-\mathbf{p}) \mathcal{D}(C)^{-1} = -H(\mathbf{p}) \quad (15)$$

is satisfied. The operation of particle-hole conjugation reverses momentum and spin but leaves position invariant [14]. For the Dirac Hamiltonian (1b) in \mathbf{p} representation, the relation (15) is equivalent to the conditions $\mathcal{D}(C) \mathbf{v}^* \mathcal{D}(C)^{-1} = \mathbf{v}$ and $\mathcal{D}(C) \beta^* \mathcal{D}(C)^{-1} = -\beta$. For the Dirac Hamiltonian in (3+1)D, we have [14]

$$\mathcal{D}(C) = i \beta v_2 \equiv i \gamma^2 , \quad (16)$$

which implies $\mathcal{D}(\kappa^2) = +\mathbb{1}$.

1.2.5. Chirality and Energy Reflection Symmetry In (3+1)D, an additional matrix $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ can be defined that anticommutes with all other γ^μ . In the massless limit $m = 0$, γ^5 commutes with the Dirac Hamiltonian

$$[H_D(\mathbf{p}, m = 0), \gamma^5] = 0, \quad (17)$$

so that it becomes the generator χ for a continuous chiral symmetry [14] with $\mathcal{D}(\chi) = \gamma^5$.

In the context of random-matrix theory (RMT), the term *chiral symmetry* has departed from its original meaning in relativistic quantum mechanics [17, 18]. The chirality matrix γ^5 anticommutes also with the *Dirac operator* $D = \gamma^\mu p_\mu - m$, see Eq. (5). Except for zero modes, the eigenfunctions of D occur in pairs u_n , $\gamma^5 u_n$ with opposite eigenvalues $(\lambda_n, -\lambda_n)$. This notion [23] has been transferred to other systems. In RMT, a *chiral symmetry* [17, 18] of a Hamiltonian H is associated with a unitary operator \mathcal{M} that satisfies

$$\mathcal{D}(\mathcal{M}) H(\mathbf{p}) \mathcal{D}^{-1}(\mathcal{M}) = -H(\mathbf{p}) \quad . \quad (18)$$

This operation reverses the energy. In recent studies [9], the operator \mathcal{M} has therefore been called *energy-reflection* (ER) symmetry. In order to distinguish \mathcal{M} from the chiral symmetry (17), we will follow the latter naming convention. The operator \mathcal{M} reverses the velocity, while leaving momentum and position invariant: $\mathcal{M} \mathbf{p} = \mathbf{p}$ and $\mathcal{M} \mathbf{r} = \mathbf{r}$. For the Dirac Hamiltonian, the corresponding representation matrix $\mathcal{D}(\mathcal{M})$ must therefore anticommute with the matrices \mathbf{v} and β .

For the (unperturbed) Dirac equation (1), time reversal symmetry (12), particle-hole conjugation (15) and the energy reflection symmetry (18) are not independent symmetries. [18] We have

$$\mathcal{M} = \kappa \theta \quad (19)$$

giving

$$\mathcal{D}(\mathcal{M}) = \mathcal{D}(C) [\mathcal{D}(\mathcal{T})]^* \quad , \quad (20)$$

and explicitly in (3+1)D

$$\mathcal{D}(\mathcal{M}) = -i \gamma^0 \gamma^5 \equiv -i \beta \mathcal{D}(\chi) \quad , \quad (21)$$

which implies $\mathcal{D}(\mathcal{M}^2) = +\mathbb{1}$.

2. (2+1)-dimensional Dirac theory

In this Section, issues relating to the discrete symmetries for the (2+1)D Dirac theory are discussed. In (2+1)D spacetime, the Dirac Hamiltonian becomes a 2×2 matrix. In this case, we have only four linearly independent basis matrices that we may choose as Pauli matrices [22] σ_x , σ_y , and σ_z , as well as $\sigma_0 \equiv \mathbb{1}_{2 \times 2}$. Yet we still need to represent intrinsic angular momentum (spin) and two energy bands. The reduced

number of basis matrices thus gives rise to pathologies that prevent the consistent definition of many symmetries, except in the mass-less limit ($m = 0$).

Explicitly, we can choose a representation where

$$\mathbf{v} = (\sigma_x, \sigma_y) \quad \text{and} \quad \beta = \sigma_z . \quad (22)$$

Our representation is equivalent, but not identical, to previously discussed ones [5, 6, 8], as a look at the γ matrices shows:

$$\gamma^0 = \sigma_z , \quad \gamma^1 = i\sigma_y , \quad \gamma^2 = -i\sigma_x . \quad (23)$$

2.1. Angular Momentum and Rotations \mathcal{R} Straightforward calculation establishes that the (2+1)D version of H_D commutes with the operator of total angular momentum $J_z = L_z + S_z$, where $L_z = x p_y - y p_x$ is the z component of orbital angular momentum, and

$$S_z \equiv \frac{1}{4i} (\mathbf{v} \times \mathbf{v}) \cdot \hat{\mathbf{z}} = \frac{1}{2} \sigma_z . \quad (24)$$

Thus the intrinsic spin degree of freedom formally emerges in complete analogy to the (3+1)D case; see Eq. (4) and Ref. [4], but only its z component is a relevant quantity in (2+1)D.

Rotations \mathcal{R}_z by 2π about the z axis map the system onto itself with a minus sign

$$\mathcal{D}(\mathcal{R}_z(2\pi)) = \exp(2\pi i J_z) = -\mathbb{1} , \quad (25)$$

similar to Eq. (10), so that once again the symmetry group becomes a double group [20].

2.2. Parity \mathcal{P} In three spatial dimensions, parity \mathcal{P} is defined as space inversion $\mathbf{r} \rightarrow -\mathbf{r}$. Its interesting aspects arise from the fact that the determinant of the associated representation matrix is $\det(\mathcal{D}(\mathcal{P})) = -1$, so that parity cannot be expressed in terms of infinitesimal unitary transformations. In two spatial dimensions, the transformation $\mathbf{r} = (x, y) \rightarrow -\mathbf{r} = (-x, -y)$ is not interesting because it is equivalent to a rotation by π and the determinant of the representation matrix is $+1$. This is why parity in 2D is usually defined [5–8] as either

$$\mathbf{r} = (x, y) \rightarrow \mathcal{P}_x \mathbf{r} = (x, -y) \quad (26a)$$

or

$$\mathbf{r} = (x, y) \rightarrow \mathcal{P}_y \mathbf{r} = (-x, y) . \quad (26b)$$

The same transformation laws apply for linear momentum \mathbf{p} and velocity \mathbf{v} . Clearly, parity invariance as a property of a physical operator should be established with respect to both operations (26a) and (26b) to count as a proper symmetry. The above 2D versions of parity imply that L_z is odd under parity,

$$\mathcal{P}_x L_z = \mathcal{P}_y L_z = -L_z . \quad (27)$$

If we require that $\mathbf{v} \cdot \mathbf{p}$ is even under parity, we find

$$\mathcal{D}(\mathcal{P}_\nu) = \sigma_\nu \quad \text{with} \quad \nu = x, y \quad . \quad (28)$$

Furthermore, this implies [5] that S_z and J_z are likewise odd under parity. In contrast, β must be even under parity. As $S_z = \beta/2$ for the (2+1)D Dirac theory, it is impossible to satisfy both conditions, i.e., the mass term violates parity symmetry in (2+1)D. Only in the mass-less limit $m = 0$, H_D is invariant under parity.

We can study parity also for the $\psi_{\mathbf{p}}^\pm(\mathbf{r})$ given in Eq. (3), which are eigenfunctions for both the Dirac Hamiltonian and linear momentum $\mathbf{p} = (p_x, p_y)$. (See also Problem 3.4 in Ref. [24].) In the $m = 0$ limit and for $E^2 = p_x^2 + p_y^2 \neq 0$, we have (apart from a normalization constant)

$$\psi_{\mathbf{p}}^\pm(\mathbf{r}) \propto e^{i\mathbf{p} \cdot \mathbf{r}} \begin{pmatrix} E_\pm \\ p_x + ip_y \end{pmatrix} \quad (29)$$

If parity is defined as above it follows that

$$\mathcal{D}(\mathcal{P}_\nu) \psi_{\mathbf{p}}^\pm(\mathbf{r}) = \eta_{\mathbf{p}}^\pm \psi_{\mathcal{P}_\nu^{-1}\mathbf{p}}^\pm(\mathcal{P}_\nu^{-1}\mathbf{r}) \quad . \quad (30)$$

with a phase factor $\eta_{\mathbf{p}}^\pm$. (Of course, we have $\mathcal{P}_\nu^{-1} = \mathcal{P}_{\nu'}$.) Similar to Eq. (7), this equation illustrates the fact that we can view a symmetry transformation such as parity from two equivalent perspectives: we may regard it as a unitary transformation of the expansion coefficients (i.e., the spinors). This is equivalent to the inverse coordinate transformation, where position \mathbf{r} and momentum \mathbf{p} are mapped on the inversely transformed quantities. As expected, parity flips position vectors and momenta, while the energy is preserved. Also, the expectation value of S_z is reversed under parity.

2.3. Time reversal θ The basic requirements for TR invariance imply that β should be even under TR while S_z should be odd. As in the case of parity, the fact that both quantities are represented by σ_z in the (2+1)D situation makes it impossible to find such a TR transformation, i.e., the mass term violates both parity and TR symmetry. However, TR can be defined consistently in the massless limit: the usual [22]

$$\theta = \mathcal{T}\mathcal{K} \quad \text{with} \quad \mathcal{D}(\mathcal{T}) = i\sigma_y \quad , \quad (31)$$

where σ_y is the imaginary Pauli matrix, implies

$$\theta \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \mathcal{D}(\mathcal{T}) \begin{pmatrix} a_1^* \\ a_2^* \end{pmatrix} = \begin{pmatrix} a_2^* \\ -a_1^* \end{pmatrix} \quad (32)$$

for arbitrary spinors $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $\mathcal{D}(\theta^2) = -\mathbb{1}$. More explicitly, we obtain for the wave function (29)

$$\theta \psi_{\mathbf{p}}^\pm = \mathcal{D}(\mathcal{T}) \psi_{\mathbf{p}}^{\pm *} = \zeta_{\mathbf{p}}^\pm \psi_{-\mathbf{p}}^\pm \quad (33)$$

with a phase factor $\zeta_{\mathbf{p}}^\pm$, i.e., momentum is reversed, while the energy is preserved. Also, the expectation value of S_z is reversed under TR.

2.4. Particle-hole conjugation \mathcal{C} As v_x and β are real matrices in our adopted representation of the (2+1)D Dirac theory while v_y is imaginary, the matrix \mathcal{C} entering the definition (14) of particle-hole conjugation needs to commute with v_x and anticommute with both v_y and β . These conditions are satisfied by the matrix σ_x . Hence, the particle-hole-conjugation operation $\kappa = \mathcal{C}\mathcal{K}$ can be consistently defined for the (2+1)D case via the representation matrix

$$\mathcal{D}(\mathcal{C}) = \sigma_x \quad (34)$$

so that $\mathcal{D}(\kappa^2) = +\mathbb{1}$.

2.5. Energy Reflection Symmetry \mathcal{M} ER symmetry requires $\mathcal{D}(\mathcal{M})$ to anticommute with all Dirac matrices (i.e., the components of \mathbf{v} and β). No such matrix exists in (2+1)D and, hence, ER symmetry cannot be established for the Dirac model. However, $\mathcal{D}(\mathcal{M}) = \sigma_z$ satisfies the ER condition (18) in the mass-less limit with $\mathcal{D}(\mathcal{M}^2) = +\mathbb{1}$.

2.6. Chiral symmetry χ The fact that no matrix anticommutes with all three Pauli matrices, i.e., the γ^μ given in Eq. (23), also implies that no equivalent of γ^5 exists in (2+1)D [5, 6]. As a result, no chiral symmetry χ can be established, *even in the mass-less limit*.

2.7. Discussion It appears that particle-hole conjugation and spatial rotations are the only symmetries that can reasonably be considered within the (2+1)D Dirac theory. The fact that consistent parity, TR and ER operations exist only in the zero-mass limit, and that chiral symmetry is altogether impossible to define, makes this theory highly pathological from a physical point of view. As it turns out, this unsatisfactory situation can be remedied by introducing an additional flavour degree of freedom for the Dirac fermions [5]. This generalization of (2+1)D Dirac theory will be explored in the next Section.

3. Two-flavour (2+1)D Dirac theory

It is possible to construct a theory of Dirac fermions in (2+1)D that exhibits the familiar symmetries of parity and TR, even for a finite mass [5]. This theory involves four-spinor wave functions describing two flavours (2Fs) of such fermions. The Dirac operator (1b) is block-diagonal, and a possible choice for the matrix operators \mathbf{v} and β is

$$\mathbf{v} = \left(\begin{pmatrix} \sigma_x & 0 \\ 0 & \sigma_x \end{pmatrix}, \begin{pmatrix} \sigma_y & 0 \\ 0 & \sigma_y \end{pmatrix} \right) \quad \text{and} \quad \beta = \begin{pmatrix} \sigma_z & 0 \\ 0 & -\sigma_z \end{pmatrix}. \quad (35)$$

Such a flavour-symmetric representation was adopted, e.g., in Refs. [25–28] to describe the electronic degrees of freedom near the two valleys in graphene. Other formulations found in the literature [5–9, 29] are unitarily equivalent to the one used here. According to Eq. (24), the spin operator S_z emerges as

$$S_z = \frac{1}{2} \begin{pmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{pmatrix}. \quad (36)$$

Thus, in contrast to the single-flavour case, β and S_z are *not* proportional to each other in the 2F (2+1)D Dirac theory. For completeness, we also give the γ matrices:

$$\gamma^0 = \beta, \quad \gamma^1 = \begin{pmatrix} i\sigma_y & 0 \\ 0 & -i\sigma_y \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} -i\sigma_x & 0 \\ 0 & i\sigma_x \end{pmatrix}. \quad (37)$$

3.1. Symmetries Rotations are again generated by $J_z = L_z + S_z$, which is a conserved quantity. Hence, as in the single-flavour case, a 2π rotation is equivalent to $-\mathbb{1}$, which is a signature of half-integer spin.

We define parity, TR and ER as symmetry transformations that connect the flavour subspaces. For each of the parity transformations $\mathcal{P}_{x,y}$, two inequivalent representation matrices $\mathcal{D}_\pm(\mathcal{P}_\nu)$ can be realised,

$$\mathcal{D}_\pm(\mathcal{P}_\nu) = \begin{pmatrix} 0 & \sigma_\nu \\ \pm\sigma_\nu & 0 \end{pmatrix}, \quad (38)$$

which are distinguished by the property $\mathcal{D}_\pm(\mathcal{P}_\nu^2) = \pm\mathbb{1}$. Similarly, the anti-unitary operator $\theta = \mathcal{T} \mathcal{K}$ with representation matrices

$$\mathcal{D}_\pm(\mathcal{T}) = \begin{pmatrix} 0 & -i\sigma_y \\ \pm i\sigma_y & 0 \end{pmatrix} \quad (39)$$

keeps the 2F (2+1)D Dirac Hamiltonian invariant while reversing velocity, momentum and spin. Action on a general state yields

$$\mathcal{D}_\pm(\theta) \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} -a_4^* \\ a_3^* \\ \pm a_2^* \\ \mp a_1^* \end{pmatrix}, \quad (40)$$

which implies $\mathcal{D}_\pm(\theta^2) = \pm\mathbb{1}$. Thus it is possible [30] that Dirac particles in 2F (2+1)D behave under TR like spinless particles [having $\mathcal{D}(\theta^2) = +\mathbb{1}$] as an alternative to the expected spin-1/2-behaviour [where $\mathcal{D}(\theta^2) = -\mathbb{1}$].

The conditions associated with invariance under particle-hole conjugation $\kappa = C \mathcal{K}$ are satisfied by

$$\mathcal{D}_\pm(C) = \begin{pmatrix} \sigma_x & 0 \\ 0 & \mp\sigma_x \end{pmatrix}. \quad (41)$$

Unlike the TR and chirality operations, particle-hole conjugation preserves the flavour degree of freedom so that we get $\mathcal{D}_\pm(\kappa^2) = +\mathbb{1}$.

Finally, there exist two inequivalent representation matrices for ER symmetry

$$\mathcal{D}_\pm(\mathcal{M}) = \begin{pmatrix} 0 & \sigma_z \\ \pm\sigma_z & 0 \end{pmatrix}. \quad (42)$$

They both anticommute with all Dirac matrices in the 2F (2+1)D model, satisfy the relation $\mathcal{D}_\varepsilon(\mathcal{M}) = \mathcal{D}_\pm(C) \mathcal{D}_{\pm\varepsilon}(\mathcal{T})^* [\varepsilon = \pm 1; \text{ cf. also Eq. (20)}]$ and, similarly to the case

TABLE 1. Summary of 2F (2+1)D operators given in compact notation where Pauli matrices σ_j and τ_j act in Dirac and flavour space, respectively, and $\sigma_0 = \tau_0 = \mathbb{1}_{2 \times 2}$.

v_x	v_y	$\beta \equiv \gamma^0$	γ^1	γ^2	$2S_z$
$\sigma_x \otimes \tau_0$	$\sigma_y \otimes \tau_0$	$\sigma_z \otimes \tau_z$	$i\sigma_y \otimes \tau_z$	$-i\sigma_x \otimes \tau_z$	$\sigma_z \otimes \tau_0$
$\mathcal{D}_+(\mathcal{P}_\nu)$	$\mathcal{D}_-(\mathcal{P}_\nu)$	$\mathcal{D}_+(\mathcal{T})$	$\mathcal{D}_-(\mathcal{T})$	$\mathcal{D}_+(\mathcal{C})$	$\mathcal{D}_-(\mathcal{C})$
$\sigma_\nu \otimes \tau_x$	$i\sigma_\nu \otimes \tau_y$	$\sigma_y \otimes \tau_y$	$-i\sigma_y \otimes \tau_x$	$\sigma_x \otimes \tau_z$	$\sigma_x \otimes \tau_0$
$\mathcal{D}_+(\mathcal{M})$	$\mathcal{D}_-(\mathcal{M})$	$\mathcal{D}_+(\chi)$	$\mathcal{D}_-(\chi)$		
$\sigma_z \otimes \tau_x$	$i\sigma_z \otimes \tau_y$	$\sigma_0 \otimes \tau_x$	$i\sigma_0 \otimes \tau_y$		

of the other flavour-coupling discrete symmetry transformations, are distinguishable by the sign of their squares: $\mathcal{D}_\pm(\mathcal{M}^2) = \pm \mathbb{1}$.

A continuous chiral symmetry also exists in 2F (2+1)D models [6, 9]. In our representation, the matrices

$$\mathcal{D}_\pm(\chi) = \begin{pmatrix} 0 & \sigma_0 \\ \pm\sigma_0 & 0 \end{pmatrix} \quad (43)$$

anticommute with β and commute with $v_{x,y}$, thus satisfying the condition for the generator of a chiral symmetry. Again, the two matrices are distinguishable by the sign of their squares: $\mathcal{D}_\pm(\chi^2) = \pm \mathbb{1}$. It is possible to express the $\mathcal{D}_\pm(\chi)$ in terms of other Dirac operators and the ER transformations,

$$\mathcal{D}_\pm(\chi) = -i v_x v_y \mathcal{D}_\pm(\mathcal{M}) = \beta \mathcal{D}_\mp(\mathcal{M}) \quad , \quad (44)$$

which mirrors Eq. (21).

3.2. Discussion For reference and to facilitate easier comparison with the literature, we provide expressions of relevant operators for the 2F (2+1)D Dirac model using a compact notation (with Pauli matrices σ_j and τ_j acting in Dirac and flavour space, respectively) in Table 1.

The existence of two possible realisations for each of the symmetries $\mathcal{S} = \mathcal{P}_\nu, \mathcal{T}, \mathcal{C}, \mathcal{M}$, and χ seems puzzling, especially for many of those transformations where the squares of the two realisations have opposite signs. However, for any specific physical realisation of a 2F (2+1)D Dirac system, the transformation properties of the basis functions will uniquely determine for each \mathcal{S} which of the two operators $\mathcal{D}_\pm(\mathcal{S})$ corresponds to the actual symmetry in this system. For fermionic (spin-1/2) particles, TR will be represented by [22] $\mathcal{D}_-(\mathcal{T})$. In contrast, $\mathcal{D}_+(\mathcal{T})$ applies to electrons in graphene [27] whose low-energy band structure (neglecting the real spin) realises a 2F (2+1)D Dirac model, with the valley index being the flavour degree of freedom [1].

In previous discussions [6] of chiral symmetry in 2F (2+1)D Dirac systems, the existence of two possible representations $\mathcal{D}_\pm(\chi)$ was seen to imply the existence of

a $U(2)$ symmetry with generators $\{\mathbb{1}, \mathcal{D}_+(\chi), \mathcal{D}_-(\chi), \mathcal{D}_+(\chi)\mathcal{D}_-(\chi)\}$. Considering also the other pairs of representation matrices $\mathcal{D}_\pm(\mathcal{S})$, we find more generally

$$\mathcal{D}_+(\mathcal{S}) \mathcal{D}_-(\mathcal{S}) = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}. \quad (45)$$

for each symmetry \mathcal{S} . Hence the product of the two possible representation matrices for each symmetry acts differently in the subspaces associated with the two flavours. Therefore, in situations where the two flavours are physically indistinguishable (which is the case, e.g., in an ideal sheet of graphene), the two representation matrices $\mathcal{D}_\pm(\mathcal{S})$ cannot be interpreted as distinct discrete symmetries of this system.

4. (1+1)-dimensional Dirac theory

For comparison, we briefly discuss the (1+1)D Dirac model where $\mathbf{p} \equiv p$ in H_D given by Eq. (1b). For definiteness we choose $\mathbf{v} = \sigma_x$ and $\beta = \sigma_z$. Parity requires

$$\mathcal{D}(\mathcal{P}) = \sigma_z \quad (46)$$

with $\mathcal{D}(\mathcal{P}^2) = +\mathbb{1}$. For TR, we find $\theta = \mathcal{T} \mathcal{K}$ with

$$\mathcal{D}(\mathcal{T}) = \sigma_z. \quad (47)$$

Thus we have $\mathcal{D}(\theta^2) = +\mathbb{1}$, consistent with the fact that a Dirac particle in (1+1)D is a spinless object [4]. Particle-hole conjugation is realised by $\kappa = C \mathcal{K}$ where

$$\mathcal{D}(C) = \sigma_x \quad (48)$$

with $\mathcal{D}(\kappa^2) = +\mathbb{1}$. The ER transformation with

$$\mathcal{D}(\mathcal{M}) = -i \sigma_y \quad (49)$$

and $\mathcal{D}(\mathcal{M}^2) = +\mathbb{1}$ satisfies the condition (18), and $\mathcal{D}(\chi) = \sigma_x$ defines a chiral symmetry.

5. Conclusions & Outlook

We investigated basic properties of discrete symmetry operations for low-dimensional Dirac models. An additional flavour degree of freedom is needed to introduce the usual discrete symmetries known from Dirac particles in three spatial dimensions. For the two-flavour (2+1)-dimensional Dirac model two inequivalent representation matrices exist for each symmetry operation. With the exception of particle-hole conjugation, the two representation matrices for each symmetry are distinguishable by the sign of their squares.

Symmetry analyses such as that presented in this work enable the classification of perturbations to the (2+1)-dimensional Dirac model [18], making it possible to draw conclusions about how the perturbations affect the spectrum and, ultimately, the physical properties of condensed-matter realisations such as graphene sheets. The origin of such perturbations could be disorder [25, 31, 32] or \mathbf{p} -dependent corrections to the Hamiltonian reflecting the lower crystal symmetry of, e.g., the graphene lattice [21].

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